

Equations of Motion for the Quantum Characteristic Functions

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In this paper, we derive equations of motion for the normal-order, the symmetric-order and the antinormal-order quantum characteristic functions, applicable for general Hamiltonian systems. We do this by utilizing the ‘characteristic form’ of both quantum states and Hamiltonians. The equations of motion we derive here are rather simple in form and in essence, and as such have a number of attractive features. As we shall see, our approach enables the descriptions of quantum and classical time evolutions in one unified language. It allows for a direct comparison between quantum and classical dynamics, providing insight into the relations between quantum and classical behavior, while also revealing a smooth transition between quantum and classical time evolutions. In particular, the $\hbar \rightarrow 0$ limit of the quantum equations of motion instantly recovers their classical counterpart. We also argue that the derived equations may prove to be very useful in numerical simulations.

I. INTRODUCTION

Quantum characteristic functions play an important role in quantum optics, serving as alternative descriptions of quantum states – descriptions which are advantageous over other equivalent formalisms in certain situations, for instance, in the evaluation of expectation values of operator moments [1]. Quantum characteristic functions are also very useful in describing the dynamics of open systems such as the damped harmonic oscillator, the quantum Brownian particle, two-level atoms and lasers (for reviews, see for example [2, 3, 4]). For these kinds of systems, one usually reformulates the appropriate master equation (which involves Hilbert space operators) in terms of characteristic functions, to obtain a differential Fokker-Planck equation, whose methods of solution have been very well studied [1]. Recently, quantum characteristic functions have been shown to be appealing from an experimental point of view as well, providing valuable tools for highly sensitive measurements [5].

While the use of characteristic functions for the description of quantum states has a long history, to the best of our knowledge, a general equation of motion (EOM) to describe their dynamics has not yet been formulated [6]. This is with the exception of [7], where a general EOM for somewhat modified quantum characteristic functions has been constructed, using a group-theoretical approach.

In this paper, we suggest a novel derivation of the EOMs for the normal-order, symmetric-order and antinormal-order quantum characteristic functions which makes use of the characteristic form of both quantum states and Hamiltonians. As we shall see, this approach provides several attractive features: Firstly, the derivation of the equations is both very general and straightforward. It leads to EOMs which are rather simple in form;

they involve neither Hilbert-space operators nor infinite sums (as opposed to, e.g., the EOM for the Wigner function), making them very appealing both for analytical calculations as well as for practical purposes such as numerical simulations.

Moreover, as we shall see, in their final form the EOMs turn out to be very similar to their classical counterpart. This property avails insight into the relation between quantum and classical time evolutions. Furthermore, the formulation of quantum and classical evolutions in a unified language enables the evaluation of the classical limit in a straightforward manner. These issues are discussed in greater detail later on.

The paper is organized as follows. In Sec. II we recall the definition of the quantum characteristic functions, and review some of their fundamental properties. In Sec. III, we carry out the derivation of the EOMs, and in Sec. IV the relation of the equations to their classical analogue is discussed. We shall conclude with a few comments.

II. THE QUANTUM CHARACTERISTIC FUNCTIONS

Three kinds of quantum characteristic functions have been widely used in the literature [8]. These are the normal-order, symmetric-order, and antinormal-order functions, defined, respectively, by the following mappings of density matrices $\hat{\rho}$ to complex-valued functions [1]

$$C^{(n)}(\lambda, \mu) = \text{Tr}[\hat{\rho} e^{i\tilde{\xi}\hat{a}^\dagger} e^{i\xi\hat{a}}], \quad (2.1a)$$

$$C^{(s)}(\lambda, \mu) = \text{Tr}[\hat{\rho} e^{i\xi\hat{a} + i\tilde{\xi}\hat{a}^\dagger}], \quad (2.1b)$$

$$C^{(a)}(\lambda, \mu) = \text{Tr}[\hat{\rho} e^{i\xi\hat{a}} e^{i\tilde{\xi}\hat{a}^\dagger}]. \quad (2.1c)$$

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Here, ξ is a complex variable defined by the real coordinates (λ, μ) as

$$\xi = \sqrt{\frac{\hbar}{2\omega}} (\lambda - i\omega\mu), \quad (2.2)$$

and the operators \hat{a} and \hat{a}^\dagger are annihilation and creation operators related to the usual position and momentum operators by

$$\begin{aligned} \hat{x} &= \sqrt{\frac{\hbar}{2\omega}} (\hat{a}^\dagger + \hat{a}), \\ \hat{p} &= i\sqrt{\frac{\hbar\omega}{2}} (\hat{a}^\dagger - \hat{a}). \end{aligned} \quad (2.3)$$

The relations between the three characteristic functions are given by

$$C^{(n)}(\lambda, \mu) = e^{\frac{1}{2}|\xi|^2} C^{(s)}(\lambda, \mu) = e^{|\xi|^2} C^{(a)}(\lambda, \mu), \quad (2.4)$$

and they satisfy accordingly the following bounds

$$|C^{(n)}(\lambda, \mu)| \leq e^{\frac{1}{2}|\xi|^2}, \quad (2.5a)$$

$$|C^{(s)}(\lambda, \mu)| \leq 1, \quad (2.5b)$$

$$|C^{(a)}(\lambda, \mu)| \leq e^{-\frac{1}{2}|\xi|^2}. \quad (2.5c)$$

It will be useful for our purpose to also recall the inverse of (2.1a), which maps the normal-order characteristic functions to density matrices

$$\hat{\rho} = \frac{1}{4\pi^2} \int dx dp \int d\lambda d\mu e^{-i(\lambda x + \mu p)} C^{(n)}(\lambda, \mu) |\alpha\rangle\langle\alpha|. \quad (2.6)$$

Here, $|\alpha\rangle$ is a coherent state, where the complex variable α is defined by the canonical coordinates (x, p) via

$$\alpha = \frac{1}{\sqrt{2\hbar\omega}} (\omega x + ip). \quad (2.7)$$

We also note that the normal-order, the symmetric-order, and the antinormal-order characteristic functions (2.1) may also serve as the definitions of the Glauber-Sudarshan P -representation [9], the Wigner function [10], and the Husimi Q quasi-probability distribution [11], respectively, by means of a Fourier transform.

III. EQUATIONS OF MOTION FOR THE QUANTUM CHARACTERISTIC FUNCTIONS

The time evolution of an arbitrary quantum state $\hat{\rho}$ whose dynamics is governed by some Hamiltonian \hat{H} is given by the von Neumann equation

$$\frac{\partial}{\partial t} \hat{\rho} = \frac{i}{\hbar} (\hat{\rho} \hat{H} - \hat{H} \hat{\rho}). \quad (3.1)$$

In what follows, we rewrite this equation in terms of the normal-order characteristic function $C^{(n)}(\lambda, \mu)$. We shall first consider one-dimensional systems; the generalization to multiple dimensions as well as the formulation of the equation in terms of the symmetric and antinormal characteristic functions will be discussed later.

Using Eq. (2.6), the left-hand-side of the von Neumann equation is simply rewritten as

$$\begin{aligned} \frac{\partial}{\partial t} \hat{\rho} &= \\ \frac{1}{4\pi^2} \int dx dp \int d\lambda d\mu e^{-i(\lambda x + \mu p)} \frac{\partial C^{(n)}(\lambda, \mu)}{\partial t} |\alpha\rangle\langle\alpha|. \end{aligned} \quad (3.2)$$

As for the right-hand-side, we make use of the fact that quantum observables, specifically Hamiltonians, may also be formally represented in characteristic form [3]. In the normal-order case, this representation reads

$$\hat{H} = \frac{1}{2\pi\hbar} \int dx dp \int d\lambda d\mu e^{-i(\lambda x + \mu p)} H^{(n)}(\lambda, \mu) |\alpha\rangle\langle\alpha|, \quad (3.3)$$

with the inverse transformation given by

$$2\pi H^{(n)}(\lambda, \mu) = \hbar \text{Tr}[\hat{H} e^{i\bar{\xi}\hat{a}^\dagger} e^{i\xi\hat{a}}]. \quad (3.4)$$

In terms of $C^{(n)}(\lambda, \mu)$ and $H^{(n)}(\lambda, \mu)$, the product $\hat{\rho}\hat{H}$ in the von Neumann equation thus becomes

$$\hat{\rho}\hat{H} = \frac{1}{8\pi^3\hbar} \int d\lambda_1 d\mu_1 d\lambda_2 d\mu_2 \int dx_1 dp_1 dx_2 dp_2 C^{(n)}(\lambda_1, \mu_1) H^{(n)}(\lambda_2, \mu_2) e^{-i(\lambda_1 x_1 + \mu_1 p_1 + \lambda_2 x_2 + \mu_2 p_2)} \langle\alpha_1|\alpha_2\rangle |\alpha_1\rangle\langle\alpha_2|. \quad (3.5)$$

Next, we identify the characteristic representation of $\hat{\rho}\hat{H}$, by expressing the operator $|\alpha_1\rangle\langle\alpha_2|$ in Eq. (3.5), in a

coherent state diagonal form according to the identity

[12]

$$|\alpha_1\rangle\langle\alpha_2| = \int d^2\alpha \text{Tr} [|\alpha_1\rangle\langle\alpha_2|\hat{\Delta}(\alpha, \bar{\alpha})] |\alpha\rangle\langle\alpha|, \quad (3.6)$$

where

$$\hat{\Delta}(\alpha, \bar{\alpha}) = \frac{1}{\pi^2} \int d^2\alpha' e^{-\alpha'(\bar{\alpha}-\hat{a}^\dagger)} e^{\bar{\alpha}'(\alpha-\hat{a})}. \quad (3.7)$$

Integrating over the internal variables, Eq. (3.5) simplifies to

$$\hat{\rho}\hat{H} = \int dx dp \int d\lambda d\mu C_{\rho H}^{(n)}(\lambda, \mu) e^{-i(\lambda x + \mu p)} |\alpha\rangle\langle\alpha|, \quad (3.8)$$

with $C_{\rho H}^{(n)}(\lambda, \mu)$ – the normal-order characteristic function of the product $\hat{\rho}\hat{H}$ – given by

$$C_{\rho H}^{(n)}(\lambda, \mu) = \int d\lambda' d\mu' C^{(n)}(\lambda', \mu') H^{(n)}(\lambda - \lambda', \mu - \mu') \times e^{\frac{\hbar}{2\omega}(\lambda' + i\omega\mu')(\lambda - \lambda' - i\omega(\mu - \mu'))}. \quad (3.9)$$

Similarly, we find that the normal-order characteristic function of the product $\hat{H}\hat{\rho}$ is given by

$$C_{H\rho}^{(n)}(\lambda, \mu) = \int d\lambda' d\mu' C^{(n)}(\lambda', \mu') H^{(n)}(\lambda - \lambda', \mu - \mu') \times e^{\frac{\hbar}{2\omega}(\lambda' - i\omega\mu')(\lambda - \lambda' + i\omega(\mu - \mu'))}. \quad (3.10)$$

Substituting the expressions for $\hat{\rho}\hat{H}$ and $\hat{H}\hat{\rho}$ into the von Neumann equation and equating the integrands on each side, we arrive at the final form

$$\frac{\partial}{\partial t} C^{(n)}(\lambda, \mu) = \int d\lambda' d\mu' K^{(n)}(\lambda, \mu, \lambda', \mu') \times H^{(n)}(\lambda - \lambda', \mu - \mu') C^{(n)}(\lambda', \mu'), \quad (3.11)$$

with the ‘normal-order kernel’ $K^{(n)}(\lambda, \mu, \lambda', \mu')$ identified as

$$K^{(n)}(\lambda, \mu, \lambda', \mu') = \frac{2}{\hbar} e^{\frac{\hbar}{2\omega}[\lambda'(\lambda - \lambda') + \omega^2\mu'(\mu - \mu')]} \sin \frac{\hbar}{2}(\lambda\mu' - \mu\lambda'). \quad (3.12)$$

Eq.(3.11) is the EOM for one-dimensional normal-order characteristic functions.

Of course, in cases where the dynamics of the system under consideration is governed by a Hamiltonian consisting of a linear combination of products of annihilation and creation operators, Eq. (3.11) can be further simplified. This is due to the fact the characteristic representation of Hamiltonians of the above form are linear combinations of delta functions and derivatives thereof [3], so the right-hand-side of (3.11) may be immediately integrated to produce a linear partial differential equation for $C^{(n)}(\lambda, \mu)$, to be further solved by the usual methods.

The EOMs for the symmetric-order $C^{(s)}(\lambda, \mu)$ and the antinormal $C^{(a)}(\lambda, \mu)$ are readily obtained by making use of the relations (2.4) both for $C^{(n)}(\lambda, \mu)$ and for $H^{(n)}(\lambda, \mu)$. The resultant equations have the same form as Eq. (3.11), but with different kernels. The antinormal-order kernel is

$$K^{(a)}(\lambda, \mu, \lambda', \mu') = \frac{2}{\hbar} e^{-\frac{\hbar}{2\omega}[\lambda'(\lambda - \lambda') + \omega^2\mu'(\mu - \mu')]} \sin \frac{\hbar}{2}(\lambda\mu' - \mu\lambda'), \quad (3.13)$$

whereas the symmetric-order kernel turns out to be particularly simple and reads

$$K^{(s)}(\lambda, \mu, \lambda', \mu') = \frac{2}{\hbar} \sin \frac{\hbar}{2}(\lambda\mu' - \mu\lambda'). \quad (3.14)$$

The simple form of the symmetric kernel enables further simplification of the ‘symmetric-order’ EOM (Eq. (3.11) with $(n) \rightarrow (s)$). It allows one to Fourier transform the EOM and to obtain an EOM for the Wigner function $W(x, p)$ – the Fourier transform of $C^{(s)}(\lambda, \mu)$. Expanding the kernel $K^{(s)}$ in a Taylor series in \hbar , the right-hand-side of the symmetric EOM becomes

$$\sum_{m=0}^{\infty} b_m \int d\lambda' d\mu' \left[\mu'(\lambda - \lambda') - \lambda'(\mu - \mu') \right]^{2m+1} \times H^{(s)}(\lambda - \lambda', \mu - \mu') C^{(s)}(\lambda', \mu'), \quad (3.15)$$

with $b_m = (-1)^m \frac{(\hbar/2)^{2m}}{(2m+1)!}$. Further expanding the bracketed term above in a binomial series, and Fourier transforming the resultant convolutions term by term, the EOM becomes

$$\frac{\partial}{\partial t} W(x, p) = \sum_{m=0}^{\infty} \sum_{k=0}^{2m+1} b_m (-1)^k \binom{2m+1}{k} i^{2(2m+1)} \left(\frac{\partial^k}{\partial x^k} \frac{\partial^{2m+1-k}}{\partial p^{2m+1-k}} W(x, p) \right) \left(\frac{\partial^{2m+1-k}}{\partial x^{2m+1-k}} \frac{\partial^k}{\partial p^k} \tilde{H}^{(s)}(x, p) \right), \quad (3.16)$$

where $\tilde{H}^{(s)}(x, p)$ is the Fourier transform of $H^{(s)}(\lambda, \mu)$.

Eq. (3.16) is thus a generalized EOM for the Wigner

function under general Hamiltonian dynamics.

This EOM may be written in a compact form with the help of the following notation:

$$\frac{\partial^{(1)}}{\partial p} \frac{\partial^{(2)}}{\partial x} AB \equiv \left(\frac{\partial}{\partial p} A \right) \left(\frac{\partial}{\partial x} B \right), \quad (3.17)$$

where A and B are arbitrary functions of the canonical variables. Via this definition, Eq. (3.16) simplifies to

$$\begin{aligned} & \frac{\partial}{\partial t} W(x, p) \\ &= \sum_{m=0}^{\infty} b_m \left(\frac{\partial^{(1)}}{\partial p} \frac{\partial^{(2)}}{\partial x} - \frac{\partial^{(1)}}{\partial x} \frac{\partial^{(2)}}{\partial p} \right)^{2m+1} W(x, p) \tilde{H}^{(s)}(x, p) \\ &= \frac{2}{\hbar} \sin \frac{\hbar}{2} \left(\frac{\partial^{(1)}}{\partial p} \frac{\partial^{(2)}}{\partial x} - \frac{\partial^{(1)}}{\partial x} \frac{\partial^{(2)}}{\partial p} \right) W(x, p) \tilde{H}^{(s)}(x, p). \end{aligned} \quad (3.18)$$

In the special case where $\tilde{H}^{(s)}(x, p) = p^2/2 + V(x)$, Eq. (3.18) reduces to the well-known equation for the time evolution of the Wigner function [13]

$$\begin{aligned} \frac{\partial}{\partial t} W(x, p) &= \left(\frac{dV(x)}{dx} \frac{\partial}{\partial p} - p \frac{\partial}{\partial x} \right. \\ &\quad \left. + \sum_{m=0}^{\infty} \frac{d^{2m+1}V(x)}{dx^{2m+1}} \frac{\partial^{2m+1}}{\partial p^{2m+1}} \right) W(x, p). \end{aligned} \quad (3.19)$$

The generalization of the EOMs derived above to N dimensions is straightforward. Now, points in phase space are represented by vector pairs $(\mathbf{x}, \mathbf{p}) = (x_1, \dots, x_N, p_1, \dots, p_N)$, and their Fourier counterparts by $(\boldsymbol{\lambda}, \boldsymbol{\mu}) = (\lambda_1, \dots, \lambda_N, \mu_1, \dots, \mu_N)$. In an exact analogy with the one-dimensional case, it is an easy matter to check that the EOM for the N -dimensional characteristic function $C^{(n)}(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is

$$\begin{aligned} \frac{\partial C^{(n)}(\boldsymbol{\lambda}, \boldsymbol{\mu})}{\partial t} &= \int d\boldsymbol{\lambda}' d\boldsymbol{\mu}' K^{(n)}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\lambda}', \boldsymbol{\mu}') \\ &\quad \times H^{(n)}(\boldsymbol{\lambda} - \boldsymbol{\lambda}', \boldsymbol{\mu} - \boldsymbol{\mu}') C^{(n)}(\boldsymbol{\lambda}', \boldsymbol{\mu}'), \end{aligned} \quad (3.20)$$

with

$$\begin{aligned} & K^{(n)}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\lambda}', \boldsymbol{\mu}') \\ &= \frac{2}{\hbar} e^{\frac{\hbar}{2\omega} [\boldsymbol{\lambda}'(\boldsymbol{\lambda} - \boldsymbol{\lambda}') + \omega^2 \boldsymbol{\mu}'(\boldsymbol{\mu} - \boldsymbol{\mu}')] } \sin \frac{\hbar}{2} (\boldsymbol{\lambda} \boldsymbol{\mu}' - \boldsymbol{\mu} \boldsymbol{\lambda}'), \end{aligned} \quad (3.21)$$

while the EOMs for the symmetric-order $C^{(s)}(\boldsymbol{\lambda}, \boldsymbol{\mu})$ and the antinormal $C^{(a)}(\boldsymbol{\lambda}, \boldsymbol{\mu})$ are obtained by making use of the multi-dimensional form of relations (2.4):

$$\begin{aligned} C^{(n)}(\boldsymbol{\lambda}, \boldsymbol{\mu}) &= e^{\frac{\hbar}{4\omega} (\boldsymbol{\lambda}^2 + \omega^2 \boldsymbol{\mu}^2)} C^{(s)}(\boldsymbol{\lambda}, \boldsymbol{\mu}) \\ &= e^{\frac{\hbar}{2\omega} (\boldsymbol{\lambda}^2 + \omega^2 \boldsymbol{\mu}^2)} C^{(a)}(\boldsymbol{\lambda}, \boldsymbol{\mu}), \end{aligned} \quad (3.22)$$

with similar relations for $H^{(n)}(\boldsymbol{\lambda}, \boldsymbol{\mu})$. The antinormal-order kernel then becomes

$$\begin{aligned} & K^{(a)}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\lambda}', \boldsymbol{\mu}') \\ &= \frac{2}{\hbar} e^{-\frac{\hbar}{2\omega} [\boldsymbol{\lambda}'(\boldsymbol{\lambda} - \boldsymbol{\lambda}') + \omega^2 \boldsymbol{\mu}'(\boldsymbol{\mu} - \boldsymbol{\mu}')] } \sin \frac{\hbar}{2} (\boldsymbol{\lambda} \boldsymbol{\mu}' - \boldsymbol{\mu} \boldsymbol{\lambda}'), \end{aligned} \quad (3.23)$$

whereas the symmetric-order kernel is

$$K^{(s)}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\lambda}', \boldsymbol{\mu}') = \frac{2}{\hbar} \sin \frac{\hbar}{2} (\boldsymbol{\lambda} \boldsymbol{\mu}' - \boldsymbol{\mu} \boldsymbol{\lambda}'). \quad (3.24)$$

IV. RELATION TO CLASSICAL DYNAMICS

Next, we show that the von Neumann equation, in its form (3.20), enables a direct comparison between quantum and classical time evolutions. To see this, let us write down the corresponding classical EOM, namely the Liouville equation [14], in terms of classical characteristic functions. As with the quantum equation, we start off with a general one-dimensional system and generalize to multiple dimensions later.

The time evolution of a statistical distribution $P(x, p)$ evolving in time under a Hamiltonian $H(x, p)$ is given by the Liouville equation

$$\frac{\partial P}{\partial t} = \frac{\partial P}{\partial x} \frac{\partial H}{\partial p} - \frac{\partial P}{\partial p} \frac{\partial H}{\partial x}. \quad (4.1)$$

In terms of the characteristic functions of $P(x, p)$ and $H(x, p)$, defined by the Fourier transforms

$$P(x, p) = \int d\lambda d\mu C^{(c)}(\lambda, \mu) e^{-i(\lambda x + \mu p)}, \quad (4.2a)$$

$$H(x, p) = \int d\lambda d\mu H^{(c)}(\lambda, \mu) e^{-i(\lambda x + \mu p)}, \quad (4.2b)$$

the Liouville equation becomes

$$\begin{aligned} \frac{\partial C^{(c)}(\lambda, \mu)}{\partial t} &= \int d\lambda' d\mu' K^{(c)}(\lambda, \mu, \lambda', \mu') \\ &\quad \times H^{(c)}(\lambda - \lambda', \mu - \mu') C^{(c)}(\lambda', \mu'), \end{aligned} \quad (4.3)$$

where the ‘classical kernel’ $K^{(c)}(\lambda, \mu, \lambda', \mu')$ is given by

$$K^{(c)}(\lambda, \mu, \lambda', \mu') = \lambda \mu' - \mu \lambda'. \quad (4.4)$$

This can be readily verified by substituting (4.2) into (4.1). In N dimensions the equation generalizes to

$$\begin{aligned} \frac{\partial C^{(c)}(\boldsymbol{\lambda}, \boldsymbol{\mu})}{\partial t} &= \int d\boldsymbol{\lambda}' d\boldsymbol{\mu}' K^{(c)}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\lambda}', \boldsymbol{\mu}') \\ &\quad \times H^{(c)}(\boldsymbol{\lambda} - \boldsymbol{\lambda}', \boldsymbol{\mu} - \boldsymbol{\mu}') C^{(c)}(\boldsymbol{\lambda}', \boldsymbol{\mu}'), \end{aligned} \quad (4.5)$$

with

$$K^{(c)}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\lambda}', \boldsymbol{\mu}') = \boldsymbol{\lambda} \cdot \boldsymbol{\mu}' - \boldsymbol{\mu} \cdot \boldsymbol{\lambda}'. \quad (4.6)$$

Having the quantum EOM (3.20) and the classical EOM (4.5) on an equal footing, the difference between quantum and classical time evolutions becomes transparent. This difference is encapsulated in the dissimilarity between the quantum and classical kernels; this is with the understanding that in both cases the Hamiltonian characteristic function is the same.

Moreover, comparing the symmetric-order kernel with the classical kernel, we obtain the simple relation

$$K^{(s)} = \frac{2}{\hbar} \sin \frac{\hbar}{2} K^{(c)}. \quad (4.7)$$

Relation (4.7) reveals a smooth transition between quantum time evolution and classical evolution, one in which \hbar naturally plays the role of a ‘quantumness’ parameter. Specifically, in the $\hbar \rightarrow 0$ limit of $K^{(s)}$, we obtain the fully classical behavior, namely $\lim_{\hbar \rightarrow 0} K^{(s)} = K^{(c)}$. As one would expect, the $\hbar \rightarrow 0$ limit of normal-order and the antinormal-order kernels also recover the classical kernel $K^{(c)}$.

V. SUMMARY AND FURTHER REMARKS

In formulating the von Neumann equation as an integro-differential equation for each of the three quantum characteristic functions, we have obtained a ‘classical’ description for the time evolution of quantum systems, which further enabled the treatment of quantum and classical systems on an equal footing. As a bonus we have also obtained a generalized EOM for the Wigner function. In the new formulation, the difference between the quantum and classical EOMs narrows down to the dissimilarity between the respective kernels. Moreover, the correspondence between quantum and classical systems arises in this formulation very naturally; a straightforward evaluation of the $\hbar \rightarrow 0$ limit of the quantum kernels recovers the classical one. This new formulation

may thus become very useful when one is interested in contrasting the behavior of quantum systems with their parallels in the classical world, as analogies can be drawn between classical phenomena and phenomena generated by quantum dynamics. In fact, a similar strategy has recently proved to be very helpful in the context of connecting the no-broadcasting theorem with its classical counterpart [15].

We believe that the EOM presented here, may prove to be useful for practical reasons as well. The reason for that is twofold. Firstly, the quantum characteristic functions are scalar-valued and are ensured to be well-behaved (in the case of the symmetric and the antinormal functions – also bound); when numerical simulations are concerned, these properties are of utmost importance, as they allow computations to take place in a rather straightforward manner and without having to worry about divergences and singularities in the evolving systems. This is in contrast with the Glauber-Sudarshan P -representation [9] for example, which is ill-defined for some perfectly reasonable quantum states and is thus very problematic in this respect. Secondly, the equation derived here is ‘classical’ in nature; it involves neither Hilbert-space operators nor infinite sums. This property makes the equation very convenient for simulation purposes, as this type of equations is fairly simple to implement computationally.

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